2011 AP exam solutions - BC

1)

a)

speed = $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$. The velocity vector = $<4t+1,\sin\left(t^2\right)>$. At t=3, velocity vector = $<13,\sin\left(9\right)>$. So speed = $\sqrt{\left(13\right)^2+\left(\sin(9)\right)^2}$. This is about 13.00653073 .

acceleration vector= $\vec{a}(t)=<4,2t\cos\left(t^2\right)>$. $\vec{a}(3)=<4,2(3)\cos\left(3^2\right)>=<4,6\cos\left(9\right)>$

b)

$m=\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\sin\left(t^2\right)}{4t+1}$. At t = 3, $m=\frac{\sin\left(3^2\right)}{4(3)+1}=\frac{\sin(9)}{13}=0.0317014219.$

c)

$x(3)=x(0)+\int_0^3\frac{dx}{dt}dt$ and $y(3)=y(0)+\int_0^3\frac{dy}{dt}dt$

$x(3)=0+\int_0^3 4t+1\,dt=\left(2t^2+t\right)\Big|_0^3=21-0=21$ and $y(3)=-4+\int_0^3\sin\left(t^2\right)dt=-3.226437473$ (from the calculator since we can't find the antiderivative of dy/dt.

d)

In a problem like this, the total distance the particle travels is the arc length.

Arc length = $\int_{t=0}^{t=3}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_{t=0}^{t=3}\sqrt{\left(4t+1\right)^2+\left(\sin\left(t^2\right)\right)^2}\,dt=21.09119045$

I would guess 3 points for a, and 2 for each of b, c, and d ….

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2) **same as AB**

a)

To approximate the rate at which the temperature of the tea is changing at t=3.5 you need to approximate $H'(t)$.

Approximate an instantaneous rate of change with an **average** rate of change.

$$H'(3.5) \approx \frac{H(5)-H(2)}{5-2} = \frac{52-60}{3} = -\frac{8}{3}$$ degrees Celsius per minute.

b)

$\frac{1}{10}\int_0^{10} H(t)\,dt$ is the average value of H(t) over the interval [0, 10]

In the context of this problem it means **the average temperature of the pot of tea – in degrees Celsius – over the first 10 minutes (time t=0 to t=10).**

Trapezoidal Sum:

$$\frac{1}{10}\int_0^{10} H(t)\,dt \approx \frac{1}{10}\left(\frac{1}{2}(2-0)(66+60)+\frac{1}{2}(5-2)(60+52)+\frac{1}{2}(9-5)(52+44)+\frac{1}{2}(10-9)(44+43)\right)$$

degrees Celsius. And the question is answered!

c)

$$\int_0^{10} H'(t)\,dt = H(10)-H(0) = 43-66 = -23$$ degrees Celsius

This is the **total change in H(t) over the interval [0,10].** In other words, $\int_0^{10} H'(t)\,dt$ represents the total change in temperature (a drop of 23 degrees Celsius) during the first 10 minutes.

d)

temperature of the biscuits after 10 minutes:

$$100^\circ + \int_0^{10} B'(t)\,dt = 100^\circ + \int_0^{10} -13.84e^{-0.173t}\,dt = 100^\circ - 65.8172472 = 34.1827528^\circ$$ Celsius.

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The temperature of the pot of tea is 43 degrees Celsius (from the table)

The biscuits are, then, 8.817247202 degrees Celsius cooler than the pot of tea.

My guess for scoring would be:

a) 1 point b) 3 points c) 2 points d) 3 points

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3)

a)

Perimeter = sum of lengths of 4 sides (but top side is a curve so we want arc length)

$$perimeter = 1 + k + e^{2k} + \int_0^k \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 1 + k + e^{2k} + \int_0^k \sqrt{1 + \left(2e^{2x}\right)^2}\, dx$$

b)

$$Volume = \pi \int_0^k \left(e^{2x}\right)^2 dx = \pi \int_0^k e^{4x}\, dx = \frac{\pi}{4} e^{4x} \Big|_0^k = \frac{\pi}{4}\left(e^{4k} - 1\right)$$

c)

$$V = \frac{\pi}{4}\left(e^{4k} - 1\right) \Rightarrow \frac{dV}{dt} = \frac{\pi}{4}\left(4e^{4k}\frac{dk}{dt}\right) = \frac{\pi}{4}\left(4e^{4(1/2)}\left(\frac{1}{3}\right)\right) = \frac{\pi}{3}e^2$$

Possible scoring:

a) 3 points   b) 3 points   c) 3 points

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4) **same as AB**

a)

$$g(-3)=2(-3)+\int_0^{-3} f(t)dt=-6-\int_{-3}^{0} f(t)dt=-6-(\text{area of quarter circle})=-6-\frac{1}{4}\pi(3)^2$$

$$g'(x)=\left(\frac{d}{dx}\right)\left(2x+\int_0^x f(t)dt\right)=2+f(x)$$
$$g'(-3)=2+f(-3)=2$$

b)

g has an absolute maximum requires *g'* to be 0 or undefined.

$g'(x)=\left(\frac{d}{dx}\right)\left(2x+\int_0^x f(t)dt\right)=2+f(x)$. It is never undefined, and is only 0 when $f(x)=-2$. This happens only at … [slope of line segment = $\frac{-3-3}{3-0}=-2$ … so equation is: $y=-2x+3$ …. Solve $-2x+3=-2 \Rightarrow -2x=-5 \Rightarrow x=5/2$] … x = 2.5

The second derivative of *g* is the slope of that line segment (= -2); it is negative, therefore *g* is concave down, and the point at x=-2 represents a relative maximum. Now: *g* is increasing on [-4,2.5] and decreasing on [2.5,3] so the relative maximum must be the absolute maximum.

c)

$g''(x)=f'(x)$. $g'(x)=2+f(x)$. From the graph f' (and therefore g'') is not defined at x = -3 and x=0. *f* is increasing throughout [-4,0] so the concavity of *g* does not change. *g'* changes from increasing to decreasing at *x* = 0 and g'' is undefined there, so *x* = 0 is a point of inflection.

d)

average rate of change of f = $\frac{f(3)-f(-4)}{3-(-4)}=\frac{-3-(-1)}{7}=-\frac{2}{7}$

The MVT requires differentiability on the open interval (in this case (-4,3) …) so MVT does not apply. The function *f* is not differentiable at x = -3 and x=0.

Possible scoring:

a) 3 points b, c, d) 2 points each

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5) **same as AB**

a)

$m = \left.\dfrac{dW}{dt}\right|_{\substack{t=0 \\ W=1400}} = \dfrac{1}{25}(1400-300) = \dfrac{1100}{25} = 44$. We are following this tangent line for a Δx of $\dfrac{1}{4}$, so $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{1/4} = 44.$ So, $\Delta y = 11$. The approximate amount of solid waste is 1411 tons at t=.25 years

b)

$$\frac{d^2W}{dt^2} = \frac{d}{dt}\left(\frac{dW}{dt}\right) = \frac{d}{dt}\left(\frac{1}{25}(W-300)\right) = \frac{1}{25}\frac{dW}{dt} = \left(\frac{1}{25}\right)\left(\frac{1}{25}\right)(W-300).$$

At t=0 and W=1400, $\dfrac{d^2W}{dt^2}$ is positive and therefore W is not only increasing, but also concave up. Therefore our approximation is an underestimate (the tangent line is under the curve).

c)

$$\frac{dW}{dt} = \frac{1}{25}(W-300) \Rightarrow \frac{dW}{W-300} = \frac{1}{25}dt \Rightarrow \int \frac{dW}{W-300} = \int \frac{1}{25}dt$$

$So... \ln(W-300) = \dfrac{t}{25} + C \Rightarrow e^{\ln(W-300)} = e^{\frac{t}{25}+C} = e^C e^{\frac{t}{25}}$ where K is an arbitrary constant.

$W - 300 = Ke^{t/25}$

Solving for K (t=0 and W=1400) we get:

$W = 300 + 1100e^{t/25}$

Possible scoring:

a) 2 b) 2 c) 5

Note that we are given a differential equation involving $\dfrac{dW}{dt}$. This tells us that the solution (the function W) will be a function of *t* …

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6)

a)

$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$. $\sin\left(x^2\right) \approx \left(x^2\right) - \frac{\left(x^2\right)^3}{3!} + \frac{\left(x^2\right)^5}{5!} - \frac{\left(x^2\right)^7}{7!} = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$

b)

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\sin\left(x^2\right) + \cos x \approx 1 + x^2\left(1 - \frac{1}{2!}\right) + x^4\left(\frac{1}{4!}\right) + x^6\left(-\frac{1}{3!} - \frac{1}{6!}\right)$$

c)

Our approximation is good enough to find $f^{(6)}(0)$ since each term after what we have will have an *x* term (and hence will become 0 when we substitute in 0)

So … $f^{(6)}(0) = -6!\left(\frac{1}{3!} + \frac{1}{6!}\right) = -\left((4)(5)(6) + 1\right) = -121$

d)

The error bound involving the 4th degree Taylor polynomial is:

$\left|error\right| < \frac{M\left|x\right|^5}{5!}$ We estimate M from the graph and notice it's about 36. We are interested in the approximation at $x = \frac{1}{4}$. So … $\left|error\right| < \frac{36\left|\left(\frac{1}{4}\right)\right|^5}{5!} < \frac{40\left|\left(\frac{1}{2^{10}}\right)\right|}{120} = \frac{1}{(1024)(3)} = \frac{1}{3072} < \frac{1}{3000}$

Possible scoring:

a) 2 b) 3 c) 2 d) 2

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